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STABILITY OF A QUADRATIC-CUBIC-QUARTIC FUNCTIONAL EQUATION

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Abstract. In this paper, we investigate the stability of a quadratic-cubic-quartic functional equation

$$f(x+ky)+f(x-ky)-k^2f(x+y)-k^2f(x-y) - 2(1-k^2)f(x)-\frac{k^2(k^2-1)}{6}(f(2y)+2f(-y)-6f(y))=0$$

by applying the direct method in the sense of Găvruta.

1. Introduction

In this paper, let V and W be real vector spaces, Y be a real Banach space and $k \in \mathbb{R}\backslash\{0, \pm1\}$. For a given mapping $f : V \to W$ with $f(0) = 0$, we use the following abbreviations

$$\begin{aligned}
f_o(x) &:= \frac{f(x)-f(-x)}{2}, \quad f_e(x) := \frac{f(x)+f(-x)}{2},\\
Df(x,y) &:= f(x+ky)+f(x-ky)-k^2f(x+y)-k^2f(x-y)\\
&\quad - 2(1-k^2)f(x)-\frac{k^2(k^2-1)}{6}(f(2y)+2f(-y)-6f(y))\\
J_nf(x) &:= \frac{f_o(2^nx)}{8^n}+\frac{16f_e(2^nx)-f_e(2^{n+1}x)}{12\cdot 4^n}\\
&\quad -\frac{4f_e(2^nx)-f_e(2^{n+1}x)}{12\cdot 16^n},\\
J'_nf(x) &:= 8^nf_o\left(\frac{x}{2^n}\right)+\frac{4\cdot 16^n-4^n}{3}f_e\left(\frac{x}{2^n}\right)\\
&\quad -\frac{16^{n+1}-4^{n+2}}{3}f_e\left(\frac{x}{2^{n+1}}\right)
\end{aligned}$$

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for all $x, y \in V$ and all $n \in \mathbb{N} \cup \{0\}$. Gordji et al. [3] proved the stability of the quadratic-cubic-quartic functional equation $Df(x,y) = 0$ for a fixed $k \in \mathbb{Z} \backslash \{0, \pm 1\}$. For the terminology "quadratic-cubic-quartic functional equation", refer to the papers [2, 3]. This author [5] proved the stability of the functional equation $Df(x,y) = 0$ for the case $k = 2$ by using the fixed point theory, and also showed the Hyers-Ulam-Rassias stability of the quadratic-cubic-quartic functional equation

$$\begin{aligned} &f(x+ky) + f(x-ky) - k^2 f(x+y) - k^2 f(x-y) \\ &\quad + 2(k^2-1)f(x) + (k^2+k^3)f(y) + (k^2-k^3)f(-y) - 2f(ky) = 0 \end{aligned}$$

[6]. In this paper, we will prove the stability of the functional equation $Df(x,y) = 0$ in the sense of Găvruta [1] (See also [4, 7]). In other words, from the given mapping f that approximately satisfies the functional equation $Df(x,y) = 0$, we will show that the mapping F, which is the solution of the functional equation $Df(x,y) = 0$, can be constructed using the formula

$$F(x) = \lim_{n \to \infty} J_n f(x)$$

or

$$F(x) = \lim_{n \to \infty} J'_n f(x),$$

and we will prove that the mapping F is the unique solution of functional equation $Df(x,y) = 0$ near the mapping f.

2. Main theorems

The following lemma is needed to show the uniqueness of the solution mapping of the functional equation $Df(x,y) = 0$ satisfying a certain condition within the main theorem about the stability of the functional equation $Df(x,y) = 0$.

LEMMA 2.1. *If a mapping $f : V \to W$ satisfies $Df(x,y) = 0$ for all $x, y \in V$, then the equalities*

$$f(x) = J_n f(x), \tag{2.1}$$

$$f(x) = J'_n f(x) \tag{2.2}$$

hold for all $x \in V$ and all $n \in \mathbb{N} \cup \{0\}$.

Proof. Notice that if a mapping $f : V \to W$ satisfies $Df(x, y) = 0$ for all $x, y \in V$, then the equalities

$$f_e(4x) - 20f_e(2x) + 64f_e(x) = 0,$$
$$f_o(2x) - 8f_o(x) = 0$$

can be obtained from the equalities

$$f(0) = \frac{2Df(0,0)}{k^2(k^2-1)},$$

$$\begin{aligned} f_e(4x) - 20f_e(2x) + 64f_e(x) - 36f_e(0) &= \frac{12k^2 Df_e(x,x) - 12(k^2-1)Df_e(0,x)}{k^2(k^2-1)} \\ &\quad - \frac{6Df_e(0,2x) - 12Df_e(kx,x)}{k^2(k^2-1)}, \end{aligned} \tag{2.3}$$

$$f_o(2x) - 8f_o(x) = \frac{-6Df_o(0,x)}{k^2(k^2-1)} \tag{2.4}$$

for all $x \in V$. Therefore, the equality (2.1) can be derived from the equality

$$\begin{aligned} f(x) - \frac{f_o(2^n x)}{8^n} - \frac{16f_e(2^n x) + f_e(2^{n+1}x)}{12 \cdot 4^n} + \frac{4f_e(2^n x) - f_e(2^{n+1}x)}{12 \cdot 16^n} \\ = \sum_{i=0}^{n-1} \frac{2f_o(2^i x) - f_o(2^{i+1}x)}{8^{i+1}} + \sum_{i=0}^{n-1} \frac{64f_e(2^i x) - 20f_e(2^{i+1}x) + f_e(2^{i+2}x)}{12 \cdot 4^{i+1}} \\ + \sum_{i=0}^{n-1} - \frac{64f_e(2^i x) - 20f_e(2^{i+1}x) + f_e(2^{i+2}x)}{12 \cdot 16^{i+1}} \end{aligned}$$

for all $x \in V$ and $n \in \mathbb{N} \cup \{0\}$. The equality (2.2) can be easily obtained in a similar way. $\square$

In the following theorem, we can prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ by using the direct method in the sense of Găvruta.

Theorem 2.2. *Let $f : V \to Y$ be a mapping for which there exists a function $\varphi : V^2 \to [0, \infty)$ such that the inequality*

$$\|Df(x, y)\| \le \varphi(x, y) \tag{2.5}$$

holds for all $x, y \in V$ and let $f(0) = 0$. If φ has the property

$$\sum_{n=0}^{\infty} \frac{\varphi(2^n x, 2^n y)}{4^n} < \infty \tag{2.6}$$

for all $x, y \in V$, then there exists a unique solution mapping $F : V \to Y$ of the functional equation $Df(x, y) = 0$ satisfying the inequality

$$\begin{aligned} \|f(x) - F(x)\| \leq & \frac{1}{k^2|k^2-1|} \sum_{n=0}^{\infty} \left(\frac{6\varphi_e(0, 2^n x)}{8^{n+1}} + \frac{2k^2 \varphi_e(2^n x, 2^n x)}{2 \cdot 4^{n+1}} \right. \\ & \left. + \frac{2|k^2-1|\varphi_e(0, 2^n x) + \varphi_e(0, 2^{n+1} x) + 2\varphi_e(2^n kx, 2^n x)}{2 \cdot 4^{n+1}} \right) \end{aligned} \tag{2.7}$$

for all $x \in V$, where φ_e is the function defined by $\varphi_e(x, y) := \frac{\varphi(x,y)+\varphi(-x,-y)}{2}$. In particular, F is represented by the equality $F(x) = \lim_{n\to\infty} J_n f(x)$ for all $x \in V$.

Proof. From the equalities (2.3), (2.4) and (2.5), we have

$$\begin{aligned} & \|J_n f(x) - J_{n+1} f(x)\| \\ & = \left\| \frac{f_o(2^n x)}{8^n} - \frac{f_o(2^{n+1} x)}{8^{n+1}} + \left(\frac{1}{12 \cdot 4^{n+1}} - \frac{1}{12 \cdot 16^{n+1}} \right) \right. \\ & \quad \left. \times \left(64 f_e(2^n x) - 20 f_e(2^{n+1} x) + f_e(2^{n+2} x)\right) \right\| \\ & = \left\| \frac{6Df_o(0, 2^n x)}{k^2(k^2-1)8^{n+1}} \right\| + \left\| \frac{2k^2 Df_e(2^n x, 2^n x)}{k^2(k^2-1)2 \cdot 4^{n+1}} \right. \\ & \quad \left. + \frac{-2(k^2-1)Df_e(0, 2^n x) - Df_e(0, 2^{n+1} x) + 2Df_e(2^n kx, 2^n x)}{k^2(k^2-1)2 \cdot 4^{n+1}} \right\| \\ & \leq \frac{1}{k^2|k^2-1|} \left(\frac{6\varphi_e(0, 2^n x)}{8^{n+1}} + \frac{2k^2 \varphi_e(2^n x, 2^n x) + 2|k^2-1|\varphi_e(0, 2^n x)}{2 \cdot 4^{n+1}} \right. \\ & \quad \left. + \frac{\varphi_e(0, 2^{n+1} x) + 2\varphi_e(2^n kx, 2^n x)}{2 \cdot 4^{n+1}} \right) \end{aligned} \tag{2.8}$$

for all $x \in V$. It follows from (2.8) that

$$
\begin{aligned}
\|J_n f(x) - J_{n+m} f(x)\| & \\
&\le \sum_{i=n}^{n+m-1} \|J_i f(x) - J_{i+1} f(x)\| \\
&\le \frac{1}{k^2|k^2-1|} \sum_{i=n}^{n+m-1} \left(\frac{6\varphi_e(0, 2^i x)}{8^{i+1}} + \frac{2k^2 \varphi_e(2^i x, 2^i x)}{2 \cdot 4^{i+1}} \right. \\
&\quad \left. + \frac{2|k^2-1|\varphi_e(0, 2^i x) + \varphi_e(0, 2^{i+1} x) + 2\varphi_e(2^i kx, 2^i x)}{2 \cdot 4^{i+1}} \right)
\end{aligned} \tag{2.9}
$$

for all $x \in V$. In view of (2.6) and (2.9), the sequence $\{J_n f(x)\}$ is a Cauchy sequence for all $x \in V$. Since Y is complete, the sequence $\{J_n f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{n \to \infty} J_n f(x)$$

for all $x \in V$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.9) we get the inequality (2.7). With the definition of F, we easily get the equality $DF(x, y) = 0$ from the relations

$$
\begin{aligned}
&\|DF(x, y)\| \\
&= \left\| \frac{Df_o(2^n x, 2^n y)}{8^n} + \frac{16Df_e(2^n x, 2^n y) - f_e(2^{n+1} x, 2^{n+1} y)}{12 \cdot 4^n} \right. \\
&\quad \left. - \frac{4Df_e(2^n x, 2^n y) - f_e(2^{n+1} x, 2^{n+1} y)}{12 \cdot 16^n} \right\| \\
&\le \left\| \frac{Df_o(2^n x, 2^n y)}{8^n} \right\| + \left\| \frac{16Df_e(2^n x, 2^n y)}{12 \cdot 4^n} \right\| + \left\| \frac{Df_e(2^{n+1} x, 2^{n+1} y)}{12 \cdot 4^n} \right\| \\
&\le \frac{\varphi_e(2^n x, 2^n y)}{8^n} + \frac{16\varphi_e(2^n x, 2^n y)}{12 \cdot 4^n} + \frac{\varphi_e(2^{n+1} x, 2^{n+1} y)}{12 \cdot 4^n} \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}
$$

To prove the uniqueness of F, let $F' : V \to Y$ be another mapping satisfying the equality $DF'(x, y) = 0$ and the inequality (2.7). Instead of the condition (2.7), it is sufficient to show that there is a unique mapping F satisfying the simpler condition

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{2 \cdot 4^{i+1}}$$

for all $x \in V$, where

$$\Phi(2^i x) := \frac{2k^2\varphi_e(2^i x, 2^i x) + 2(k^2+4)\varphi_e(0, 2^i x) + \varphi_e(0, 2^{i+1}x) + 2\varphi_e(2^i kx, 2^i x)}{k^2|k^2-1|}.$$

By (2.1), the equality $F'(x) = J_n F'(x)$ holds for all $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned}
&\|J_n f(x) - F'(x)\| \\
&\quad = \|J_n f(x) - J_n F'(x)\| \\
&\quad \le \Bigg\| \frac{f_o(2^n x)}{8^n} + \frac{16 f_e(2^n x) - f_e(2^{n+1}x)}{12 \cdot 4^n} - \frac{4 f_e(2^n x) - f_e(2^{n+1}x)}{12 \cdot 16^n} \\
&\qquad - \frac{F_o'(2^n x)}{8^n} - \frac{16 F_e'(2^n x) - F_e'(2^{n+1}x)}{12 \cdot 4^n} + \frac{4F_e'(2^n x) - F_e'(2^{n+1}x)}{12 \cdot 16^n} \Bigg\| \\
&\quad \le \left\| \frac{f_o(2^n x)}{8^n} - \frac{F_o'(2^n x)}{8^n} \right\| + \left(\frac{16}{12 \cdot 4^n} - \frac{4}{12 \cdot 16^n} \right) \left\| f_e(2^n x) - F_e'(2^n x) \right\| \\
&\qquad + \left(\frac{1}{12 \cdot 4^n} - \frac{1}{12 \cdot 16^n} \right) \left\| f_e(2^{n+1}x) - F_e'(2^{n+1}x) \right\| \\
&\quad \le \sum_{i=0}^{\infty} \frac{\Phi(2^{i+n}x)}{2 \cdot 4^{i+1} \cdot 8^n} + \frac{4}{3} \sum_{i=0}^{\infty} \frac{\Phi(2^{i+n}x)}{2 \cdot 4^{i+1} \cdot 4^n} + \frac{4}{3} \sum_{i=0}^{\infty} \frac{\Phi(2^{i+n+1}x)}{2 \cdot 4^{i+1} \cdot 4^n} \\
&\quad \le \sum_{i=n}^{\infty} \frac{\Phi(2^i x)}{2 \cdot 4^{i+1}} + \sum_{i=n}^{\infty} \frac{\Phi(2^i x)}{4^{i+1}} + \sum_{i=n+1}^{\infty} \frac{\Phi(2^i x)}{4^{i+1}}
\end{aligned}$$

for all $x \in V$ and all $n \in \mathbb{N}$. Taking the limit in the above inequality as $n \to \infty$, we conclude that $F'(x) = \lim_{n\to\infty} J_n f(x)$ for all $x \in V$. This means that the equality $F(x) = F'(x)$ holds for all $x \in V$. □

Corollary 2.3. *Let $p \in (0, 2)$ and X be a real normed space. If $f : X \to Y$ is a mapping such that*

$$\|Df(x, y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{2.10}$$

for all $x, y \in X$, then there exists a unique solution mapping $F : X \to Y$ of the functional equation $Df(x, y) = 0$ satisfying the inequality

$$\begin{aligned}
\|f(x) - F(x)\| \le & \frac{6}{k^2|k^2-1|(8-2^p)} \\
& + \frac{4k^2 + 2|k^2-1| + 2 + 2|k|^p + 2^p}{2k^2|k^2-1|(4-2^p)}
\end{aligned} \tag{2.11}$$

for all $x \in X$.

Proof. If we put $\varphi = \theta(\|x\|^p + \|y\|^p)$ in Theorem 2.2, the inequality (2.11) is easily obtained from the inequality (2.7). $\square$

THEOREM 2.4. *Let $f : V \to Y$ be a mapping for which there exists a function $\varphi : V^2 \to [0, \infty)$ such that the inequality (2.5) holds for all $x, y \in V$ and let $f(0) = 0$. If φ has the property*

$$\sum_{n=0}^{\infty} 16^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \infty \tag{2.12}$$

for all $x, y \in V$, then there exists a unique solution mapping $F : V \to Y$ of the functional equation $Df(x, y) = 0$ satisfying the inequality

$$\|f(x) - F(x)\| \le \sum_{n=0}^{\infty} \Psi_n(x) \tag{2.13}$$

for all $x \in V$, where

$$\begin{aligned}\Psi_n(x) :=& \frac{6 \cdot 8^n}{k^2|k^2-1|}\varphi_e\left(0, \frac{x}{2^{n+1}}\right) + \frac{8 \cdot 16^n}{k^2|k^2-1|}\Bigg(2k^2\varphi_e\left(\frac{x}{2^{n+2}}, \frac{x}{2^{n+2}}\right) \\ &+ 2|k^2-1|\varphi_e\left(0, \frac{x}{2^{n+2}}\right) + \varphi_e\left(0, \frac{x}{2^{n+1}}\right) + 2\varphi_e\left(\frac{kx}{2^{n+2}}, \frac{x}{2^{n+2}}\right)\Bigg).\end{aligned}$$

In particular, F is represented by $F(x) = \lim_{n\to\infty} J'_n f(x)$ for all $x \in V$.

Proof. From the equalities (2.3), (2.4) and (2.5), we have

$$\begin{aligned}&\|J'_n f(x) - J'_{n+1} f(x)\| \\ \le& \Bigg\| 8^n\left(f_o\left(\frac{x}{2^n}\right) - 8 f_o\left(\frac{x}{2^{n+1}}\right)\right) \\ &+ \frac{(4 \cdot 16^n - 4^n)}{3}\left(f_e\left(\frac{x}{2^n}\right) - 20 f_e\left(\frac{x}{2^{n+1}}\right) + 64 f_e\left(\frac{x}{2^{n+2}}\right)\right)\Bigg\| \\ \le& \frac{6 \cdot 8^n}{k^2|k^2-1|}\left\| Df_o\left(0, \frac{x}{2^{n+1}}\right)\right\| \\ &+ \frac{4 \cdot 16^n}{3k^2|k^2-1|}\Bigg\| -6Df_e\left(0, \frac{x}{2^{n+1}}\right) + 12 Df_e\left(\frac{kx}{2^{n+2}}, \frac{x}{2^{n+2}}\right) \\ &+ 12k^2 Df_e\left(\frac{x}{2^{n+2}}, \frac{x}{2^{n+2}}\right) - 12(k^2-1)Df_e\left(0, \frac{x}{2^{n+2}}\right)\Bigg\| \\ \le& \Psi_n(x)\end{aligned} \tag{2.14}$$

for all $x \in V$. It follows from (2.14) that

$$\|J_n' f(x) - J_{n+m}' f(x)\| \leq \sum_{i=n}^{n+m-1} \Psi_i(x) \tag{2.15}$$

for all $x \in V$. In view of (2.12) and (2.15), the sequence $\{J_n' f(x)\}$ is a Cauchy sequence for all $x \in V$. Since Y is complete, the sequence $\{J_n' f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by $F(x) := \lim_{n\to\infty} J_n' f(x)$ for all $x \in V$. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.15) we get the inequality (2.13). From the definition of F, we easily get

$$\begin{aligned}
\|DF(x,y)\| =& \Big\| 8^n Df_o\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + \frac{4\cdot 16^n - 4^n}{3} Df_e\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\
&- \frac{16^{n+1} - 4^{n+2}}{3} Df_e\left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}\right) \Big\| \\
\leq& \Big\| 16^n Df_o\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \Big\| + \Big\| 16^n Df_e\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \Big\| \\
&+ \Big\| \frac{16^{n+1}}{3} Df_e\left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}\right) \Big\| \\
\leq& 16^{n+1}\varphi_e\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + 16^{n+1}\varphi_e\left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}\right) \\
\to& \, 0 \text{ as } n \to \infty,
\end{aligned}$$

which means that $DF(x,y) = 0$ for all $x, y \in V$.

To prove the uniqueness of F, let $F' : V \to Y$ be another mapping satisfying $DF'(x,y) = 0$ and (2.13). Instead of the condition (2.13), it is sufficient to show that there is a unique mapping satisfying the simpler condition

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} 16^i \Phi\left(\frac{x}{2^i}\right)$$

for all $x \in V$, where

$$\begin{aligned}
\Phi\left(\frac{x}{2^i}\right) :=& \frac{16}{k^2|k^2-1|}\Bigg(2k^2\varphi_e\left(\frac{x}{2^{i+2}}, \frac{x}{2^{i+2}}\right) \\
&+ 2|k^2-1|\varphi_e\left(0, \frac{x}{2^{i+2}}\right) + 2\varphi_e\left(0, \frac{x}{2^{i+1}}\right) + 2\varphi_e\left(\frac{kx}{2^{i+2}}, \frac{x}{2^{i+2}}\right)\Bigg).
\end{aligned}$$

By (2.2), the equality $F'(x) = J'_n F'(x)$ holds for all $x \in V$ and all $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned}
&\|J'_n f(x) - F'(x)\| \\
&=\|J'_n f(x) - J'_n F'(x)\| \\
&\leq \left\| 8^n f_o\left(\frac{x}{2^n}\right) + \frac{4\cdot 16^n - 4^n}{3} f_e\left(\frac{x}{2^n}\right) - \frac{16^{n+1} - 4^{n+2}}{3} f_e\left(\frac{x}{2^{n+1}}\right)\right. \\
&\quad \left. - 8^n F'_o\left(\frac{x}{2^n}\right) - \frac{4\cdot 16^n - 4^n}{3} F'_e\left(\frac{x}{2^n}\right) + \frac{16^{n+1} - 4^{n+2}}{3} F'_e\left(\frac{x}{2^{n+1}}\right)\right\| \\
&\leq 8^n \left\| f_o\left(\frac{x}{2^n}\right) - F'_o\left(\frac{x}{2^n}\right)\right\| + 16^{n+1}\left\| f_o\left(\frac{x}{2^{n+1}}\right) - F'_o\left(\frac{x}{2^{n+1}}\right)\right\| \\
&\quad + 16^{n+1}\left\| f_e\left(\frac{x}{2^n}\right) - F'_e\left(\frac{x}{2^n}\right)\right\| \\
&\leq \sum_{i=n}^{\infty} 16^i \Phi\left(\frac{x}{2^i}\right) + \sum_{i=n+1}^{\infty} 16^i \Phi\left(\frac{x}{2^i}\right) + \sum_{i=n}^{\infty} 16^{i+1}\Phi\left(\frac{x}{2^i}\right)
\end{aligned}$$

for all $x \in V$ and all $n \in \mathbb{N}$. Taking the limit in the above inequality as $n \to \infty$, we can conclude that $F'(x) = \lim_{n\to\infty} J'_n f(x)$ for all $x \in V$. This means that $F(x) = F'(x)$ for all $x \in V$. □

Corollary 2.5. *Let $p > 4$ be a real number and X be a real normed space. If $f : X \to Y$ is a mapping satisfying the inequality (2.10) for all $x, y \in X$, then there exists a unique solution mapping $F : X \to Y$ of the functional equation $Df(x, y) = 0$ satisfying the inequality*

$$\|f(x) - F(x)\| \leq \frac{6}{k^2|k^2-1|(2^p-8)} + \frac{8(4k^2 + 2|k^2-1| + 2|k|^p + 2 + 2^p)}{k^2|k^2-1|2^p(2^p-16)}$$

for all $x \in X$.

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